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Supersymmetric *t–J* Gaudin models and KZ equations

Mark D Gould[1], **Yao-Zhong Zhang**[1] **and Shao-You Zhao**[2]

[1] Center of Mathematical Physics, Department of Mathematics, University of Queensland, Brisbane 4072, Australia
[2] Department of Physics, Nanjing University, Nanjing 210093, People's Republic of China

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Abstract

Supersymmetric t–J Gaudin models with open boundary conditions are investigated by means of the algebraic Bethe ansatz method. Off-shell Bethe ansatz equations of the boundary Gaudin systems are derived, and used to construct and solve the KZ equations associated with $sl(2|1)^{(1)}$ superalgebra.

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1. Introduction

In the study of one-dimensional long-range interacting systems, Gaudin-type models [1] ocupied an important place, due to their role in establishing the integrability of the Seiberg–Witten theory [2, 3] and diagonalizing the BCS Hamiltonian of ultrasmall metallic grains [4–6]. They also served as a testing ground for ideas such as the functional Bethe ansatz and general procedure of separation of variables [7–10].

The t–J model was proposed in an attempt to understand high-T_c superconductivity. It is a correlated electron system with nearest-neighbour hopping (t) and anti-ferromagnetic exchange (J) of electrons. At the supersymmetric point, the model is integrable. The supersymmetric t–J model without or with a boundary has been investigated by many authors using the nested algebraic Bethe ansatz method [11–14].

The periodic supersymmetric t–J Gaudin model and its off-shell Bethe ansatz were investigated in [15, 16]. In this paper, we study the off-shell Bethe ansatz and Knizhnik–Zamolodchikov (KZ) equations of the open boundary super t–J Gaudin model.

KZ equations were first proposed as a set of differential equations satisfied by correlation functions of the Wess–Zumino–Witten models [17]. The connection between Gaudin-type magnets and the KZ equations has been studied by many authors [18–22]. We are interested in the super KZ equations associated with $sl(2|1)^{(1)}$ superalgebra. We will construct and solve these KZ equations with the help of the super t–J Gaudin models.

The outline of this paper is as follows. In section 2, we give some preliminaries on the supersymmetric t–J system. In section 3, we study KZ equations corresponding to the periodic t–J Gaudin model using the off-shell Bethe ansatz equations. Then in sections 4, we construct and diagonalize the open boundary super *t–J* Gaudin model. In section 5, we obtain solutions to the KZ equations associated with the boundary Gaudin model.

2. Preliminaries

The supersymmetric *t–J* model is described by the *R*-matrix arising from the three-dimensional representation of $sl(2|1)^{(1)}$. In the fermionic, fermionic and bosonic (FFB) grading of the representation space, the *R*-matrix is given by [23]

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & -c_-(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & c_-(\lambda) & 0 & 0 \\ 0 & -c_+(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & c_-(\lambda) & 0 \\ 0 & 0 & c_+(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_+(\lambda) & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w(\lambda) \end{pmatrix} \tag{2.1}$$

where η is a crossing parameter and

$$\begin{aligned} a(\lambda) &= 1 \qquad b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda-\eta)} \qquad w(\lambda) = \frac{\sinh(\lambda+\eta)}{\sinh(\lambda-\eta)} \\ c_+(\lambda) &= \frac{\mathrm{e}^{\lambda}\sinh(\eta)}{\sinh(\lambda-\eta)} \qquad c_-(\lambda) = \frac{\mathrm{e}^{-\lambda}\sinh(\eta)}{\sinh(\lambda-\eta)}. \end{aligned} \tag{2.2}$$

This *R*-matrix satisfies the graded Yang–Baxter equation (YBE)

$$R(\lambda-\mu)^{b_1b_2}_{a_1a_2}R(\lambda)^{c_1b_3}_{b_1a_3}R(\mu)^{c_2c_3}_{b_2b_3}(-)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} = R(\mu)^{b_2b_3}_{a_2a_3}R(\lambda)^{b_1c_3}_{a_1b_3}R(\lambda-\mu)^{c_1c_2}_{b_1b_2}(-)^{(\epsilon_{a_1}+\epsilon_{b_1})\epsilon_{b_2}} \tag{2.3}$$

where ϵ_a is the Grassmann parity: $\epsilon_a = 0$ for bosons and $\epsilon_a = 1$ for fermions. The *R*-matrix satisfies the unitarity and cross-unitarity relations,

$$\begin{aligned} &R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)\,\mathrm{id} \qquad \rho(\lambda) = -\sinh(\lambda+\eta)\sinh(\lambda-\eta) \\ &R^{st_1}_{12}(\lambda-\eta)M_1R^{st r_1}_{21}M_1^{-1} = \tilde{\rho}(\lambda)\,\mathrm{id} \qquad \tilde{\rho}(\lambda) = \sinh(\lambda)\sinh(\lambda-\eta) \end{aligned} \tag{2.4}$$

where M is a diagonal matrix diag$(\mathrm{e}^{2\eta}, 1, 1)$ and st is the super-transposition defined by

$$(A^{st})_{ij} = A_{ji}(-1)^{(\epsilon_i+1)\epsilon_j}. \tag{2.5}$$

Consider the *L*-operator

$$L_{aq}(\lambda) \equiv R_{aq}(\lambda) \tag{2.6}$$

where a represents the auxiliary space and q represents the quantum space. The *L*-operator also obeys the (graded) YBE

$$R_{12}(\lambda-\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda-\mu). \tag{2.7}$$

The tensor product is graded, namely,

$$(F\otimes G)^{bd}_{ac} = F^b_aG^d_c(-)^{(\epsilon_a+\epsilon_b)\epsilon_c}. \tag{2.8}$$

The row-to-row monodromy matrix $T(\lambda)$ is defined as the product of N operators,

$$T_a(\lambda) = L_{a1}(\lambda-z_1)L_{a2}(\lambda-z_2)\cdots L_{aN}(\lambda-z_N). \tag{2.9}$$

In matrix form,

$$\{[T(\lambda)]^{ab}\}^{\alpha_1\cdots\alpha_N}_{\beta_1\cdots\beta_N} = L_1(\lambda - z_1)^{c_1\beta_1}_{a\alpha_1} L_2(\lambda - z_2)^{c_2\beta_2}_{c_1\alpha_2} \cdots L_N(\lambda - z_N)^{b\beta_N}_{c_{N-1}\alpha_N} \times (-1)^{\sum_{j=1}^{N-1}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=j+1}^{N}\epsilon_{\alpha_i}}. \tag{2.10}$$

By repeatedly using the YBE, one can easily check that the monodromy matrix satisfies

$$R(\lambda - \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda - \mu). \tag{2.11}$$

The transfer matrix $t(\lambda)$ is defined as the supertrace of the monodromy matrix over the auxiliary space:

$$t(\lambda) = \mathrm{str}\, T(\lambda) = \sum(-1)^{\epsilon_a} T(\lambda)_{aa}. \tag{2.12}$$

Using the YBE, one can show that the transfer matrix $t(\lambda)$ constitutes a one-parameter commuting family, i.e.

$$[t(\lambda), t(\mu)] = 0. \tag{2.13}$$

Therefore, the supersymmetric t–J model is integrable.

To construct the integrable open boundary t–J model, we consider the graded reflection relation

$$R(\lambda - \mu)^{b_1b_2}_{a_1a_2} K(\lambda)^{c_1}_{b_1} R(\lambda + \mu)^{c_2d_1}_{b_2c_1} K(\mu)^{d_2}_{c_2} (-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} = K(\mu)^{b_2}_{a_2} R(\lambda + \mu)^{b_1c_2}_{a_1b_2} K(\lambda)^{c_1}_{b_1} R(\lambda - \mu)^{d_2d_1}_{c_2c_1} (-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}} \tag{2.14}$$

where $K(\lambda)$ is the reflection K-matrix. The diagonal solutions of the reflection equation were found in [14]. In the present paper, we only consider the special case in which $K(\lambda) = 1$.

Following the standard procedure, we define the double-row monodromy matrix

$$\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda). \tag{2.15}$$

Here $T(\lambda)$ is same as in the periodic case. One can check that the following relation is satisfied:

$$R(\lambda - \mu)^{b_1b_2}_{a_1a_2} \mathcal{T}(\lambda)^{c_1}_{b_1} R(\lambda + \mu)^{c_2d_1}_{b_2c_1} \mathcal{T}(\mu)^{d_2}_{c_2} (-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} = \mathcal{T}(\mu)^{b_2}_{a_2} R(\lambda + \mu)^{b_1c_2}_{a_1b_2} \mathcal{T}(\lambda)^{c_1}_{b_1} R(\lambda - \mu)^{d_2d_1}_{c_2c_1} (-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \tag{2.16}$$

The dual reflection relation reads

$$R_{12}(\mu - \lambda)K_1^+(\lambda)M_1^{-1}R_{21}(\eta - \lambda - \mu)K_2^+(\mu)M_2^{-1} = K_2^+(\mu)M_2^{-1}R_{12}(\eta - \lambda - \mu)K_1^+(\lambda)M_1^{-1}R_{21}(\mu - \lambda). \tag{2.17}$$

Solution K^+ associated with $K(\lambda) = 1$ is given by

$$K^+(\lambda) \equiv MK(-\lambda + \eta/2) = \mathrm{diag}(\mathrm{e}^{2\eta}, 1, 1). \tag{2.18}$$

Define the boundary transfer matrix,

$$t^b(\lambda) = \mathrm{str}\, K^+(\lambda)\mathcal{T}(\lambda). \tag{2.19}$$

From (2.4) and (2.16), one can show the commutativity of the transfer matrix for different λ values. Therefore, the open boundary t–J model is integrable.

3. KZ equations for the periodic t–J Gaudin model

The supersymmetric t–J Gaudin model can be obtained by taking the quasi-classical limit $\eta \to 0$ of the transfer matrix at the point $\lambda = z_j$ [24]. So we expand the transfer matrix around the point $\eta = 0$ to get

$$t(z_j) = -1 + \eta H_j + \mathcal{O}(\eta^2). \tag{3.1}$$

Then the commutation relation (2.13) implies that the Hamiltonian H_j of the periodic t–J Gaudin model satisfies $[H_j, H_k] = 0$. In terms of electron and spin operators,

$$\begin{aligned} H_j = {} & \left.\frac{\mathrm{d}t(\lambda = z_j)}{\mathrm{d}\eta}\right|_{\eta=0} \\ = {} & \sum_{k=1\neq j}^{N} \frac{1}{\sinh(z_j - z_k)} \Bigg\{ \cosh(z_j - z_k)[-n_{j,-1}n_{k,-1} - n_{j,1}n_{k,1} + (1 - n_j)(1 - n_k)] \\ & + \mathrm{e}^{-(z_j - z_k)} \left[\sum_{\sigma=\pm 1} c^\dagger_{j,\sigma}(1 - n_{j,-\sigma})c_{k,\sigma}(1 - n_{k,-\sigma}) - S^\dagger_j S_k \right] \\ & + \mathrm{e}^{z_j - z_k} \left[\sum_{\sigma=\pm 1} c_{j,\sigma}(1 - n_{j,-\sigma})c^\dagger_{k,\sigma}(1 - n_{k,-\sigma}) - S_j S^\dagger_k \right] \Bigg\} \end{aligned} \tag{3.2}$$

where $n_{j,\pm 1} = c^\dagger_{j,\pm 1}c_{j,\pm 1}$, $n_{j,+1} + n_{j,-1} = n_j$ and

$$S_j = c^\dagger_{j,1}c_{j,-1} \qquad S^\dagger_j = c^\dagger_{j,-1}c_{j,1} \qquad S^z_j = n_{j,-1} - n_{j,1}.$$

Together with the operators

$$T^z_j = 1 - n_{j,-1} \qquad Q^\dagger_{j,\sigma} = c^\dagger_{j,\sigma}(1 - n_{j,-\sigma}) \qquad Q_{j,\sigma} = c_{j,\sigma}(1 - n_{j,-\sigma})$$

S, $S^\dagger$, S^z form the $sl(2|1)$ algebra which has, among others,

$$[S^\dagger, S] = S^z \qquad \left\{Q^\dagger_1, Q_1\right\} = T^z \qquad \left\{Q^\dagger_{-1}, Q_{-1}\right\} = S^z + T^z. \tag{3.3}$$

The Hamiltonian (3.2) can be diagonalized by using the algebraic Bethe ansatz method [15, 16]. The off-shell Bethe ansatz equations can be shown to be given by

$$H_j\phi = E_j\phi + \sum_{\alpha=1}^{M} \frac{(-1)^{\alpha-1}}{\sinh(\mu_\alpha - z_j)} f_\alpha E^-_j(d_\alpha)\phi_\alpha \tag{3.4}$$

where j indicates the lattice position and $E^-_j(s)$ acts on the quantum space with $E^-_j(s) = e_{13}$ for $s = 1$ and $E^-_j(s) = e_{23}$ for $s = 2$; and

$$E_j = \sum_{\alpha=1}^{n} \coth(z_j - \mu_\alpha) - 2\sum_{k=1,\neq j}^{N} \coth(z_j - z_k) \tag{3.5}$$

$$f_\alpha = -\sum_{\gamma=1}^{m} \coth\left(\mu_\alpha - \mu^{(1)}_\gamma\right) + \sum_{k=1}^{N} \coth(\mu_\alpha - z_k) \tag{3.6}$$

$$\phi = \prod_{\alpha=1}^{n} \left(\sum_{k=1}^{N} \frac{1}{\sinh(\mu_\alpha - z_k)} E^-_k(d_\alpha) \right) |0\rangle F^{d_1 \cdots d_n} \tag{3.7}$$

$$\phi_\alpha = \prod_{\beta=1,\neq\alpha}^{n} \left(\sum_{k=1}^{N} \frac{1}{\sinh(\mu_\beta - z_k)} E^-_k(d_\beta) \right) |0\rangle F^{d_1 \cdots d_n} \tag{3.8}$$

where $F^{d_1\cdots d_n}$ is a function of the spectral parameter μ_α, and $\mu_1^{(1)}, \ldots, \mu_m^{(1)}$ satisfy the constraint

$$f_\gamma^{(1)} \equiv \sum_{\beta=1,\neq\alpha}^{n} \coth\left(\mu_\beta - \mu_\gamma^{(1)}\right) + 2 \sum_{\delta=1,\neq\gamma}^{m} \coth\left(\mu_\gamma^{(1)} - \mu_\delta^{(1)}\right) = 0. \tag{3.9}$$

In the derivation of the above off-shell Bethe ansatz equations, use has been made of the gauge transformation of the R-matrix:

$$R(\lambda) \to \mathrm{diag}(\mathrm{e}^{\lambda/2}, \mathrm{e}^{\lambda/2}, \mathrm{e}^{-\lambda/2}) \overset{s}{\otimes} 1 \cdot R(\lambda) \cdot \mathrm{diag}(\mathrm{e}^{-\lambda/2}, \mathrm{e}^{-\lambda/2}, \mathrm{e}^{\lambda/2}) \overset{s}{\otimes} 1. \tag{3.10}$$

As a set of partial differential equations, the KZ equations take the form

$$\nabla_j \Psi = 0 \qquad \text{for} \quad j = 1, 2, \ldots, N \tag{3.11}$$

where the differential operator ∇_j is defined by the Gaudin Hamiltonian H_j:

$$\nabla_j = \kappa \frac{\partial}{\partial z_j} - H_j \tag{3.12}$$

with κ being a parameter. Substituting (3.2) into (3.12), we can check

$$[\nabla_j, \nabla_k] = 0 \tag{3.13}$$

which ensures the integrability of the KZ equations.

To simplify our calculation, we make the following transformation:

$$H_j \to H_j + 2 \sum_{k=1,\neq j}^{N} \coth(z_j - z_k)$$

$$E_j \to E_j + 2 \sum_{k=1,\neq j}^{N} \coth(z_j - z_k) = \sum_{\alpha=1}^{n} \coth(z_j - \mu_\alpha).$$

Under this transformation, the form of the off-shell Bethe ansatz equations is invariant.

The function $\Psi(z)$ can be constructed by the hypergeometric function $\chi(z, \mu)$ which obeys the equations

$$\kappa \frac{\partial}{\partial z_j} \chi = E_j \chi \qquad \kappa \frac{\partial}{\partial \mu_\alpha} \chi = f_\alpha \chi \tag{3.14}$$

and the constraint $f_\gamma^{(1)} = 0$. The solution to the above equations is given by

$$\chi(z, \mu) = \prod_{\beta<\alpha} \left[\sinh\left(\mu_\alpha - \mu_\gamma^{(1)}\right)\right]^{-1/\kappa} \prod_{\alpha=1}^{n} \prod_{j=1}^{N} [\sinh(z_j - \mu_\alpha)]^{1/\kappa} \tag{3.15}$$

with μ_α satisfying the condition $f_\gamma^{(1)} = 0$. With the help of $\chi(z, \mu)$, the function $\Psi(z)$ is given by

$$\Psi(z) = \oint_C \cdots \oint_C \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_n \chi(t, z) \phi(t, z) \tag{3.16}$$

where the integration path C is a closed contour in the Riemann surface such that the integrand resumes its initial value after t_α has described it. Substituting the expressions of ∇_j and $\Psi(z)$ into (3.11), we can show that the KZ equation is satisfied.

4. Bethe ansatz for the boundary t–J Gaudin model

Similar to the periodic case, the boundary t–J Gaudin system can be obtained by expanding the boundary transfer matrix at the point $\lambda = z_j$ around $\eta = 0$:

$$t^b(\lambda = z_j) = 1 + \eta H_j^b + \mathcal{O}(\eta^2). \tag{4.1}$$

The second term on the right-hand side gives the Hamiltonian of the open boundary super t–J Gaudin model. Explicitly,

$$\begin{aligned} H_j &= \left.\frac{\mathrm{d}t(z_j)}{\mathrm{d}\eta}\right|_{\eta=0} \\ &= 2(1+3\coth(2z_j))n_j + \sum_{k=1\neq j}^{N} \frac{1}{\sinh(z_j-z_k)}\Bigg\{\cosh(z_j-z_k)[-n_{j,-1}n_{k,-1} - n_{j,1}n_{k,1} \\ &\quad + (1-n_j)(1-n_k)] + \mathrm{e}^{-(z_j-z_k)}\left[\sum_{\sigma=\pm1} c_{j,\sigma}^\dagger(1-n_{j,-\sigma})c_{k,\sigma}(1-n_{k,-\sigma}) - S_j^\dagger S_k\right] \\ &\quad + \mathrm{e}^{z_j-z_k}\left[\sum_{\sigma=\pm1} c_{j,\sigma}(1-n_{j,-\sigma})c_{k,\sigma}^\dagger(1-n_{k,-\sigma}) - S_j S_k^\dagger\right]\Bigg\} \\ &\quad + \sum_{k=1\neq j}^{N} \frac{1}{\sinh(z_j+z_k)}\Bigg\{\cosh(z_j+z_k)[-n_{j,-1}n_{k,-1} - n_{j,1}n_{k,1} + (1-n_j)(1-n_k)] \\ &\quad - \mathrm{e}^{-(z_j+z_k)}\left[\sum_{\sigma=\pm1} c_{j,\sigma}(1-n_{j,-\sigma})c_{k,\sigma}^\dagger(1-n_{k,-\sigma}) + S_j S_k^\dagger\right] \\ &\quad - \mathrm{e}^{z_j+z_k}\left[\sum_{\sigma=\pm1} c_{j,\sigma}^\dagger(1-n_{j,-\sigma})c_{k,\sigma}(1-n_{k,-\sigma}) + S_j^\dagger S_k\right]\Bigg\}. \end{aligned} \tag{4.2}$$

As the transfer matrix $t^b(\lambda)$ forms a commutation family, therefore, one can prove from (4.1) the integrability of the Hamiltonian H_j^b,

$$\left[H_j^b, H_k^b\right] = 0 \qquad \text{for} \quad j = 1, 2, \ldots, N. \tag{4.3}$$

We write the double-monodromy matrix (2.15) as

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_1(\lambda) \\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_2(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{C}_2(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{4.4}$$

Around $\eta = 0$,

$$\begin{aligned} \mathcal{T}(\lambda) &= 1 + \eta\hat{\mathcal{T}}(\lambda) + \mathcal{O}(\eta^2) \\ &= 1 + \eta \begin{pmatrix} \hat{\mathcal{A}}_{11}(\lambda) & \hat{\mathcal{A}}_{12}(\lambda) & \hat{\mathcal{B}}_1(\lambda) \\ \hat{\mathcal{A}}_{21}(\lambda) & \hat{\mathcal{A}}_{22}(\lambda) & \hat{\mathcal{B}}_2(\lambda) \\ \hat{\mathcal{C}}_1(\lambda) & \hat{\mathcal{C}}_2(\lambda) & \hat{\mathcal{D}}(\lambda) \end{pmatrix} + \mathcal{O}(\eta^2). \end{aligned} \tag{4.5}$$

Define the vacuum state

$$|0\rangle_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |0\rangle = \otimes_{k=1}^N |0\rangle_k \tag{4.6}$$

and the Bethe state

$$\phi_b = \hat{\mathcal{C}}_{d_1}(\mu_1)\hat{\mathcal{C}}_{d_2}(\mu_2)\cdots\hat{\mathcal{C}}_{d_n}(\mu_n)|0\rangle F^{d_1\cdots d_n}. \tag{4.7}$$

Applying $\hat{\mathcal{T}}(\lambda = z_j)$, $j = 1, 2, \ldots, N$, to the vacuum state (4.6), we have

$$\hat{\mathcal{B}}_a(z_j)|0\rangle = 0 \qquad \hat{\mathcal{C}}_a(z_j)|0\rangle \neq 0$$
$$\hat{\mathcal{D}}(z_j)|0\rangle = \sum_{i=1}^{N} 2(\coth(z_j - z_i) + \coth(z_j - z_i))|0\rangle \tag{4.8}$$
$$\hat{\mathcal{A}}_{ab}(\lambda)|0\rangle = \begin{cases} 0 & \text{for } \lambda = z_j, \text{ and } a \neq b \\ \sum_{i=1}^{N}(\coth(\lambda + z_i) + \coth(\lambda - z_i)) & \text{for } \lambda \neq z_j, \text{ and } a = b. \end{cases}$$

Write

$$\hat{\mathcal{A}}_{ab}(\lambda)|_{\eta=0} = \hat{\tilde{\mathcal{A}}}_{ab}(\lambda)|_{\eta=0} + \delta_{ab}\frac{1}{\sinh(2\lambda)}\mathcal{D}(\lambda)_{\eta=0} \tag{4.9}$$

where

$$\hat{\tilde{\mathcal{A}}}_{ab}(\lambda)|0\rangle = \sum_{i=1}^{N}(\coth(\lambda + z_i) + \coth(\lambda - z_i)) - \frac{\delta_{ab}}{\sinh(2\lambda)}\mathcal{D}(\lambda)_{\eta=0}. \tag{4.10}$$

Then

$$\begin{aligned} H_j^b &= \frac{\mathrm{d}}{\mathrm{d}\eta}(K^+\mathcal{T}(z_j))_{\eta=0} \\ &= -\hat{\mathcal{A}}_{aa}(z_j) + \hat{\mathcal{D}}(z_j) - \left(k_a^+\right)'_{\eta=0}\mathcal{A}(z_j)_{\eta=0} - U\mathcal{D}(\lambda)_{\eta=0} \end{aligned} \tag{4.11}$$

where $a = 1, 2$ and $U = 2/\sinh(2z_j)$. The last term in (4.11) corresponds to the boundary condition.

We now find commutation relations between $\hat{\tilde{\mathcal{A}}}_{ab}(\lambda)$, $\hat{\mathcal{D}}(\lambda)$ and $\hat{\mathcal{C}}_{d_i}(\mu)$. After a tedious but direct computation, we get

$$\hat{\mathcal{C}}_{d_1}(\mu_1)\hat{\mathcal{C}}_{d_2}(\mu_2) = -\hat{\mathcal{C}}_{c_2}(\mu_2)\hat{\mathcal{C}}_{c_1}(\mu_1) \tag{4.12}$$

$$\begin{aligned} \hat{\mathcal{D}}(z_j)\hat{\mathcal{C}}_d(\mu) &= \hat{\mathcal{C}}_d(\mu)\hat{\mathcal{D}}(z_j) - \frac{\sinh(2z_j)}{\sinh(z_j - \mu)\sinh(z_j + \mu)}\hat{\mathcal{C}}_d(\mu)D(z_j)|_{\eta=0} \\ &\quad + \frac{1}{\sinh(z_j - \mu)}(-E_j^-(d)\hat{\mathcal{D}}(\mu) + \hat{\mathcal{C}}_d(z_j)\mathcal{D}(\mu)|_{\eta=0}) \\ &\quad - \frac{1}{\sinh(z_j + \mu)}(-E_j^-(d)\hat{\tilde{\mathcal{A}}}_{bd}(\mu) + \hat{\mathcal{C}}_b(z_j)\tilde{\mathcal{A}}_{bd}(\mu)|_{\eta=0}) \\ &\quad + \frac{2\coth(2\mu)}{\sinh(z_j - \mu)}E_j^-(d)\mathcal{D}(\mu)|_{\eta=0} - \frac{2\cosh(z_j + \mu)}{\sinh^2(z_j + \mu)}E_j^-(b)\tilde{\mathcal{A}}_{bd}(\mu)|_{\eta=0} \end{aligned} \tag{4.13}$$

$$\begin{aligned} \hat{\tilde{\mathcal{A}}}_{a_1d_1}(z_j)\hat{\mathcal{C}}_{d_2}(\mu) &= \hat{\mathcal{C}}_{d_2}(\mu)\hat{\tilde{\mathcal{A}}}_{a_1d_1}(z_j) \\ &\quad + \left(r_{12}(z_j + \mu + \eta)_{a_1c_2}^{c_1b_2} r_{21}(z_j - \mu)_{b_1b_2}^{d_1d_2}\right)'_{\eta=0}\hat{\mathcal{C}}_{c_2}(\mu)\tilde{\mathcal{A}}_{c_1b_1}(\lambda)|_{\eta=0} \\ &\quad + \frac{1}{\sinh(z_j - \mu)}\delta_{a_1b_2}\delta_{b_1d_1}\left(-E_j^-(b_1)\hat{\tilde{\mathcal{A}}}_{a_1d_2}(\mu) + \hat{\mathcal{C}}_{d_1}(z_j)\tilde{\mathcal{A}}_{a_1d_2}(\mu)|_{\eta=0}\right) \\ &\quad - \frac{1}{\sinh(z_j + \mu)}\delta_{a_1d_2}\delta_{b_2d_1}\left(-E_j^-(b_2)\hat{\mathcal{D}}(\mu) + \hat{\mathcal{C}}_{b_2}(z_j)\mathcal{D}(\mu)|_{\eta=0}\right) \\ &\quad - \left(\frac{\sinh(\eta)r_{12}(2z_j + \eta)_{a_1b_1}^{b_2d_1}}{\sinh(z_j - \mu)}\right)''_{\eta=0} E_j^-(b_1)\tilde{\mathcal{A}}_{b_2d_2}(\mu)|_{\eta=0} \\ &\quad + \left(\frac{\sin(2\mu)\sin(\eta)r_{12}(2z_j + \eta)_{a_1b_2}^{d_2d_1}}{\sinh(z_j + \mu + \eta)\sinh(2\mu + \eta)}\right)''_{\eta=0} E_j^-(b_2)\mathcal{D}(\mu)|_{\eta=0} \end{aligned} \tag{4.14}$$

$$\mathcal{D}(z_j)\hat{\mathcal{C}}_d(\mu)_{\eta=0} = \hat{\mathcal{C}}_d(\mu)\mathcal{D}(z_j)_{\eta=0} + \frac{1}{\sinh(z_j-\mu)}E_j^-(d)\mathcal{D}(\mu)_{\eta=0} - \frac{1}{\sinh(z_j+\mu)}E_j^-(b)\tilde{\mathcal{A}}_{bd}(\mu)_{\eta=0} \tag{4.15}$$

$$\tilde{\mathcal{A}}_{a_1d_1}(z_j)\hat{\mathcal{C}}_{d_2}(\mu)_{\eta=0} = \hat{\mathcal{C}}_{d_2}(\mu)\tilde{\mathcal{A}}_{a_1d_1}(z_j)_{\eta=0} + \frac{1}{\sinh(z_j-\mu)}E_j^-(d_1)\tilde{\mathcal{A}}_{a_1d_2}(\mu)_{\eta=0} - \frac{1}{\sinh(z_j+\mu)}\delta_{a_1d_2}E_j^-(d_1)\mathcal{D}(\mu)_{\eta=0} \tag{4.16}$$

where the r-matrix $r(\lambda)$ is defined by

$$r(\lambda) = \frac{1}{\sinh(\lambda)}\begin{pmatrix} \sinh(\lambda-\eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & -\mathrm{e}^{-\lambda}\sinh(\eta) & 0 \\ 0 & -\mathrm{e}^{\lambda}\sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda-\eta) \end{pmatrix}. \tag{4.17}$$

Then, applying (4.11) to the Bethe state and using the above commutation relations repeatedly, we obtain the off-shell Bethe ansatz equations

$$H_j^b\phi^b = E_j^b\phi^b - \sum_{\alpha=1}^{n} W^b(\mu_\alpha, z_j) f_\alpha^b E_j^-\phi_\alpha^b \tag{4.18}$$

where

$$\phi^b = \prod_{\alpha=1}^{n}\left(\sum_{k=1}^{N}\frac{2\sinh(\mu_\alpha)\cosh(z_k)}{\sinh(\mu_\alpha-z_k)\sinh(\mu_\alpha+z_k)}E_k^-(d_\alpha)\right)|0\rangle F^{d_1\cdots d_n} \tag{4.19}$$

$$\phi_\alpha^b = \prod_{\beta=1,\neq\alpha}^{n}\left(\sum_{k=1}^{N}\frac{2\sinh(\mu_\beta)\cosh(z_k)}{\sinh(\mu_\beta-z_k)\sinh(\mu_\beta+z_k)}E_k^-(d_\beta)\right)|0\rangle F^{d_1\cdots d_n} \tag{4.20}$$

$$E_j^b = \frac{2}{\sinh(z_j)} - 2\sum_{k=1,\neq j}^{N}[\coth(z_j+z_k)+\coth(z_j-z_k)] + \sum_{\alpha=1}^{n}[\coth(z_j+\mu_\alpha)+\coth(z_j-\mu_\alpha)] \tag{4.21}$$

$$f_\alpha^b = -\frac{2}{\sinh(2\mu_\alpha)} - \sum_{\beta=1,\neq\alpha}^{n}(\coth(\mu_\alpha-\mu_\beta)+\coth(\mu_\alpha+\mu_\beta)) + \sum_{k=1}^{N}(\coth(\mu_\alpha-z_k)+\coth(\mu_\alpha+z_k)) + \Lambda_b^{(1)}(\mu_\alpha) \tag{4.22}$$

$$W^b(\mu_\alpha, z_j) = \frac{(-1)^{\alpha-1}2\sinh(z_j)\cosh(\mu_\alpha)}{\sinh(z_j+\mu_\alpha)\sinh(z_j-\mu_\alpha)}. \tag{4.23}$$

Here $\Lambda_b^{(1)}$ is the eigenvalue of the nested transfer matrix

$$\begin{aligned}\hat{t}_b^{(1)}(\lambda) &= \frac{\mathrm{d}}{\mathrm{d}\eta}\big[\mathrm{str}\, K^{(1)+}r(\lambda+\mu_1+\eta)_{ac_1}^{a_1e_1}r(\lambda+\mu_2+\eta)_{a_1c_2}^{a_2e_2}\cdots r(\lambda+\mu_n+\eta)_{a_{n-1}c_n}^{a_ne_n} \\ &\quad K^{(1)}r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\cdots r_{21}(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}r_{21}(\lambda-\mu_n)_{b_1e_1}^{b_nd_1}\big] \\ &\equiv \frac{\mathrm{d}}{\mathrm{d}\eta}\,\mathrm{str}\, K^{(1)+}\mathcal{T}^{(1)}(\lambda)\end{aligned} \tag{4.24}$$

where

$$K^{(1)+} = \mathrm{diag}(e^{2\eta}, 1) \qquad K^{(1)} = (1 - \sinh(\eta)/\sinh(2\lambda + \eta)) \cdot \mathrm{id}.$$

The nested eigenvalue $\Lambda_b^{(1)}$ can be obtained using the nested Bethe ansatz. This is done as follows. Investigating the nested r-matrix (4.17), one finds that it satisfies the unitarity and cross-unitarity relations

$$\begin{aligned} &r_{12}(\lambda)r_{21}(-\lambda) = -\sinh(\lambda+\eta)\sinh(\lambda+\eta)\cdot\mathrm{id} \\ &r_{12}^{st_1}(2\eta-\lambda)r_{21}^{st_2}(\lambda) = \sinh(\lambda)\sinh(2\lambda-\eta)\cdot\mathrm{id} \end{aligned} \tag{4.25}$$

and the YBE

$$r_{12}(\lambda-\mu)T_1^{(1)}(\lambda)T_2^{(1)}(\mu) = T_2^{(1)}(\mu)T_1^{(1)}(\lambda)r_{12}(\lambda-\mu) \tag{4.26}$$

where $T^{(1)}(\lambda)$ is defined by

$$T^{(1)}(\lambda) \equiv r_m(\lambda)\cdots r_2(\lambda)r_1(\lambda). \tag{4.27}$$

We also have boundary reflection and dual relection equations

$$r_{12}(\lambda-\mu)K_1^{(1)}(\lambda)r_{21}(\lambda+\mu)K_2^{(1)}(\mu) = K_2^{(1)}(\mu)r_{12}(\lambda+\mu)K_1^{(1)}(\lambda)r_{21}(\lambda-\mu) \tag{4.28}$$

$$\begin{aligned} &r_{12}(\mu-\lambda)K_1^{(1)^+}(\lambda)M_1^{-1}r_{21}(2\eta-\lambda-\mu)K_2^{(1)^+}(\mu)M_2^{-1} \\ &\quad = K_2^{(1)^+}(\mu)M_2^{-1}r_{12}(2\eta-\lambda-\mu)K_1^{(1)^+}(\lambda)M_1^{-1}r_{21}(\mu-\lambda). \end{aligned} \tag{4.29}$$

One can check $K^{(1)} = 2\cosh(\eta+\lambda)\sinh(\lambda)/\sinh(2\lambda+\eta)$ and $K^{(1)^+} = \mathrm{diag}(e^{2\eta}, 1)$ are solutions to the first and second equations, respectively.

Define the double-row monodromy matrix for the open boundary system

$$\begin{aligned} \mathcal{T}^{(1)}(\lambda) &\equiv T^{(1)}(\tilde{\lambda})K^{(1)}(\lambda)T^{(1)^{-1}}(-\tilde{\lambda}) \\ &= \begin{pmatrix} \mathcal{A}^{(1)}(\lambda) & \mathcal{B}^{(1)}(\lambda) \\ \mathcal{C}^{(1)}(\lambda) & \mathcal{D}^{(1)}(\lambda) \end{pmatrix} \end{aligned} \tag{4.30}$$

where $T^{(1)}$ and $T^{(1)^{-1}}$ are defined by

$$\begin{aligned} T^{(1)}_{aa_n}(\tilde{\lambda})^{e_1\cdots e_n}_{c_1\cdots c_n} &= r(\tilde{\lambda}+\tilde{\mu}_1)^{a_1e_1}_{ac_1}r(\tilde{\lambda}+\tilde{\mu}_2)^{a_2e_2}_{a_1c_2}\cdots r(\tilde{\lambda}+\tilde{\mu}_n)^{a_ne_n}_{a_{n-1}c_n} \\ &= L_1^{(1)}(\tilde{\lambda}+\tilde{\mu}_1)L_2^{(1)}(\tilde{\lambda}+\tilde{\mu}_2)\cdots L_n^{(1)}(\tilde{\lambda}+\tilde{\mu}_n) \end{aligned} \tag{4.31}$$

$$\begin{aligned} T^{(1)^{-1}}(\tilde{\lambda}) &= r_{21}(\tilde{\lambda}-\tilde{\mu}_n)^{b_{n-1}d_n}_{b_ne_n}\cdots r(\tilde{\lambda}-\tilde{\mu}_2)^{b_1d_2}_{b_2e_2}r(\tilde{\lambda}-\tilde{\mu}_1)^{ad_1}_{b_1e_1} \\ &= L_n^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_n)\cdots L_2^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_2)L_1^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_1) \end{aligned} \tag{4.32}$$

respectively, with $L^{(1)}(\lambda) \equiv r(\lambda)$. Let $\tilde{\lambda} = \lambda + \eta/2$, $\tilde{\mu} = \mu - \eta/2$. Then the above formulae coincide with those appeared in (4.24).

The double-row monodromy matrix satisfies the reflection equation

$$r_{12}(\lambda-\mu)\mathcal{T}_1^{(1)}(\lambda)r_{21}(\lambda+\mu)\mathcal{T}_2^{(1)}(\mu) = \mathcal{T}_2^{(1)}(\mu)r_{12}(\lambda+\mu)\mathcal{T}_1^{(1)}(\lambda)r_{21}(\lambda-\mu). \tag{4.33}$$

Thus, we can define the transfer matrix as

$$t_b^{(1)}(\lambda) = \mathrm{str}\, K^{(1)^+}\mathcal{T}^{(1)}(\lambda). \tag{4.34}$$

Around $\eta = 0$, we have the expansions

$$\mathcal{T}^{(1)}(\lambda) = 1 + \eta \begin{pmatrix} \hat{\mathcal{A}}^{(1)}(\lambda) & \hat{\mathcal{B}}^{(1)}(\lambda) \\ \hat{\mathcal{C}}^{(1)}(\lambda) & \hat{\mathcal{D}}^{(1)}(\lambda) \end{pmatrix}_{\eta=0} + \mathcal{O}(\eta^2) \tag{4.35}$$

$$t_b^{(1)}(\lambda) = -2 + \eta\hat{t}_b^{(1)}(\lambda)_{\eta=0} + \mathcal{O}(\eta^2). \tag{4.36}$$

Write

$$\hat{\mathcal{A}}^{(1)}(\lambda)|_{\eta=0} = \hat{\tilde{\mathcal{A}}}^{(1)}(\lambda)|_{\eta=0} - \frac{e^{-2\lambda}}{\sinh(2\lambda)}\mathcal{D}^{(1)}(\lambda)|_{\eta=0}. \tag{4.37}$$

Then we find the commutation relations between $\hat{t}_b^{(1)}(\lambda)$ and $\hat{\mathcal{C}}^{(1)}(\mu^{(1)})$:

$$\hat{\mathcal{C}}^{(1)}(\lambda)\hat{\mathcal{C}}^{(1)}(\mu^{(1)}) = \hat{\mathcal{C}}^{(1)}(\mu^{(1)})\hat{\mathcal{C}}^{(1)}(\lambda) \tag{4.38}$$

$$\begin{aligned}\hat{\mathcal{D}}^{(1)}(\lambda)\hat{\mathcal{C}}^{(1)}(\mu^{(1)}) &= \hat{\mathcal{C}}^{(1)}(\mu^{(1)})\hat{\mathcal{D}}^{(1)}(\lambda) + \frac{\sinh(2\lambda)}{\sinh(\lambda+\mu^{(1)})\sinh(\lambda-\mu^{(1)})}\hat{\mathcal{C}}^{(1)}(\mu^{(1)})\mathcal{D}^{(1)}(\lambda)_{\eta=0}\\ &\quad - \frac{e^{(\lambda-\mu^{(1)})}}{\sinh(\lambda-\mu^{(1)})}(\mathcal{C}^{(1)}(\lambda)\hat{\mathcal{D}}^{(1)}(\mu^{(1)}) + \hat{\mathcal{C}}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu^{(1)}))_{\eta=0}\\ &\quad + \frac{e^{(\lambda+\mu^{(1)})}}{\sinh(\lambda+\mu^{(1)})}(\mathcal{C}^{(1)}(\lambda)\hat{\tilde{\mathcal{A}}}^{(1)}(\mu^{(1)}) + \hat{\mathcal{C}}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu^{(1)}))_{\eta=0}\end{aligned} \tag{4.39}$$

$$\begin{aligned}\hat{\tilde{\mathcal{A}}}^{(1)}(\lambda)\hat{\mathcal{C}}^{(1)}(\mu^{(1)}) &= \hat{\mathcal{C}}^{(1)}(\mu^{(1)})\hat{\tilde{\mathcal{A}}}^{(1)}(\lambda) - \frac{\sinh(2\lambda)}{\sinh(\lambda+\mu^{(1)})\sinh(\lambda-\mu^{(1)})}\hat{\mathcal{C}}^{(1)}(\mu^{(1)})\tilde{\mathcal{A}}^{(1)}(\lambda)_{\eta=0}\\ &\quad - \frac{e^{-(\lambda+\mu^{(1)})}}{\sinh(\lambda+\mu^{(1)})}(\mathcal{C}^{(1)}(\lambda)\hat{\mathcal{D}}^{(1)}(\mu) + \hat{\mathcal{C}}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu))_{\eta=0}\\ &\quad + \frac{e^{-(\lambda-\mu^{(1)})}}{\sinh(\lambda-\mu^{(1)})}(\mathcal{C}^{(1)}(\lambda)\hat{\tilde{\mathcal{A}}}^{(1)}(\mu^{(1)}) + \hat{\mathcal{C}}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu^{(1)}))_{\eta=0}\end{aligned} \tag{4.40}$$

$$\begin{aligned}\mathcal{D}^{(1)}(\lambda)\hat{\mathcal{C}}(\mu^{(1)})_{\eta=0} &= \hat{\mathcal{C}}(\mu^{(1)})\mathcal{D}^{(1)}(\lambda)_{\eta=0} + \frac{e^{\lambda-\mu^{(1)}}}{\sinh(\lambda-\mu^{(1)})}\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu^{(1)})_{\eta=0}\\ &\quad - \frac{e^{\lambda+\mu^{(1)}}}{\sinh(\lambda-\mu^{(1)})}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu^{(1)})_{\eta=0}\end{aligned} \tag{4.41}$$

$$\begin{aligned}\tilde{\mathcal{A}}^{(1)}(\lambda)\hat{\mathcal{C}}(\mu)_{\eta=0} &= \hat{\mathcal{C}}(\mu)\tilde{\mathcal{A}}^{(1)}(\lambda)_{\eta=0} + \frac{e^{-(\lambda-\mu)}}{\sinh(\lambda-\mu)}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu)_{\eta=0}\\ &\quad - \frac{e^{-(\lambda+\mu)}}{\sinh(\lambda-\mu)}\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu)_{\eta=0}.\end{aligned} \tag{4.42}$$

Substituting the expressions of $K^{(1)}(\lambda)$ and $K^{(1)^+}(\lambda)$ into the nested transfer matrix, one obtains the transfer matrix $\hat{t}_b^{(1)}(\tilde{\lambda})$ of the super t–J Gaudin model

$$\begin{aligned}\hat{t}_b^{(1)}(\tilde{\lambda}) &\equiv \frac{d}{d\eta}\,\mathrm{str}\begin{pmatrix} e^{2\eta} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathcal{A}^{(1)}(\tilde{\lambda}) & \mathcal{B}^{(1)}(\tilde{\lambda})\\ \mathcal{C}^{(1)}(\tilde{\lambda}) & \mathcal{D}^{(1)}(\tilde{\lambda})\end{pmatrix}_{\eta=0}\\ &= -\hat{\mathcal{A}}^{(1)}(\lambda) - \hat{\mathcal{D}}^{(1)}(\lambda) - 2\mathcal{A}^{(1)}(\tilde{\lambda})_{\eta=0}\\ &= -\hat{\tilde{\mathcal{A}}}^{(1)}(\lambda) - \hat{\mathcal{D}}^{(1)}(\lambda) - 2\tilde{\mathcal{A}}^{(1)}(\tilde{\lambda})_{\eta=0} - \frac{e^{-2\tilde{\lambda}}}{\sinh(2\tilde{\lambda})}\mathcal{D}^{(1)}(\tilde{\lambda})_{\eta=0}.\end{aligned} \tag{4.43}$$

Define the vacuum state for the nested open boundary system

$$|0\rangle^{(1)} = \otimes_{k=1}^{n}|0\rangle_k^{(1)} \tag{4.44}$$

with $|0\rangle_k^{(1)} \equiv \binom{0}{1}$. Then, applying the elements of $\hat{\mathcal{T}}(\lambda)$ to the vacuum, we obtain

$$\hat{\mathcal{B}}^{(1)}(\lambda)|0\rangle = 0 \qquad \hat{\mathcal{C}}^{(1)}(\lambda)|0\rangle \neq 0 \tag{4.45}$$

$$\hat{\mathcal{D}}^{(1)}(\lambda)|0\rangle = -\left(\frac{1}{\sinh(2\lambda)} + \sum_{i=1}^{n}(\coth(\lambda+\mu_i)+\coth(\lambda-\mu_i)\right)|0\rangle^{(1)}$$
$$\lambda \neq \mu_k \qquad (k=1,2,\ldots,m). \tag{4.46}$$

Using the nested YBE (4.26) and the transformation (4.37), we obtain

$$\hat{\tilde{\mathcal{A}}}^{(1)}(\lambda)|0\rangle^{(1)} = -\frac{\mathrm{e}^{-\lambda}}{\sinh(\lambda)}|0\rangle^{(1)}. \tag{4.47}$$

The eigenvalues of the nested open boundary Gaudin system can be obtained by applying $\hat{t}_b^{(1)}(\lambda)$ to the eigenvector

$$\phi_b^{(1)} = \hat{\mathcal{C}}^{(1)}\big(\mu_1^{(1)}\big)\hat{\mathcal{C}}^{(1)}\big(\mu_2^{(1)}\big)\cdots\hat{\mathcal{C}}^{(1)}\big(\mu_m^{(1)}\big)|0\rangle. \tag{4.48}$$

The result is

$$\Lambda^{(1)}(\tilde{\lambda}) = \frac{\mathrm{e}^{(-\lambda)}}{\sinh(\lambda)} + \sum_{i=1}^{n}(\coth(\lambda+\tilde{\mu}_i)+\coth(\lambda-\mu_i))$$
$$-\sum_{l=1}^{m}\left(\coth\left(\lambda+\mu_l^{(1)}\right)+\coth\left(\lambda-\mu_l^{(1)}\right)\right) \tag{4.49}$$

where $\mu_l^{(1)}$ satisfy the constraints

$$f_\gamma^{(1)} = \sum_{i=1}^{n}\left(\coth\left(\mu_j^{(1)}+\mu_i\right)+\coth\left(\mu_j^{(1)}-\mu_i\right)\right)$$
$$-2\sum_{l=1,\neq j}^{m}\left(\coth\left(\mu_j^{(1)}+\mu_l^{(1)}\right)+\coth\left(\mu_j^{(1)}-\mu_l^{(1)}\right)\right)$$
$$=0. \tag{4.50}$$

Substituting the nested eigenvalue into (4.22), we obtain the Bethe ansatz equations for the boundary t–J Gaudin model

$$f_\alpha^b = -\frac{\mathrm{e}^{-\mu_\alpha}}{\cosh(\mu_\alpha)} + \sum_{k=1}^{N}(\coth(\mu_\alpha-z_k)+\coth(\mu_\alpha+z_k))$$
$$-\sum_{\gamma=1}^{m}\left(\coth\left(\mu_\alpha-\mu_\gamma^{(1)}\right)+\coth\left(\mu_\alpha+\mu_\gamma^{(1)}\right)\right)$$
$$=0. \tag{4.51}$$

5. Super KZ equation in the boundary case

As in the periodic case, the KZ equations are

$$\nabla_j\Psi = 0 \qquad \nabla_j = \kappa\frac{\partial}{\partial z_j} - H_j^b \qquad j=1,2,\ldots,N \tag{5.1}$$

but now H_j^b is the Hamiltonian of the boundary t–J Gaudin model. We make the transformation

$$H_j \to H_j - 2/\sinh(2z_j) + 2\sum_{k=1,\neq j}^{N}[\coth(z_j+z_k)+\coth(z_j-z_k)]$$

$$E_j^b \to E_j^b - 2/\sinh(2z_j) + 2\sum_{k=1,\neq j}^{N}[\coth(z_j+z_k)+\coth(z_j-z_k)]$$
$$= \sum_{\alpha=1}^{n}[\coth(z_j+\mu_\alpha)+\coth(z_j-\mu_\alpha)].$$

This transformation leaves invariant the form of the off-shell Bethe ansatz equations.

To construct $\Psi(z)$, we introduce a hypergeometric function $\chi(z,\mu)$ which satisfies the following equations

$$\kappa\frac{\partial}{\partial z_j}\chi = E_j^b\chi \qquad \kappa\frac{\partial}{\partial\mu_\alpha}\chi = f_\alpha^b\chi \tag{5.2}$$

and the constraint $f_\gamma^{b^{(1)}} = 0$. Solving these two equations, one gets

$$\chi(z,\mu) = \prod_{\alpha=1}^{n}(1+\mathrm{e}^{-2\mu_\alpha})^{1/\kappa}\prod_{\alpha=1}^{n}\left[\sinh\left(\mu_\alpha-\mu_\gamma^{(1)}\right)\sinh\left(\mu_\alpha-\mu_\gamma^{(1)}\right)\right]^{-1/\kappa}$$
$$\times\prod_{\alpha=1}^{n}\prod_{j=1}^{N}[\sinh(z_j+\mu_\alpha)\sinh(z_j-\mu_\alpha)]^{1/\kappa} \tag{5.3}$$

where $\mu_\gamma^{(1)}$ satisfies the nested Bethe ansatz equation $f_\gamma^{b^{(1)}} = 0$. With the help of $\chi(z,\mu)$, the function $\Psi(z)$ is given by

$$\Psi(z) = \oint_C\cdots\oint_C \mathrm{d}\mu_1\cdots\mathrm{d}\mu_M\,\chi(t,z)\phi(t,z) \tag{5.4}$$

where the integration path C is a closed contour in the Riemann surface such that the integrand resumes its initial value after t_α has described it. Substituting the expressions of ∇_j and $\Psi(z)$ into (3.11), we can show that the KZ equation is satisfied. The proof is as follows

$$\kappa\frac{\partial}{\partial z_j}\Psi(z) = \oint_C\cdots\oint_C \mathrm{d}\mu_1\cdots\mathrm{d}\mu_M\left(\kappa\frac{\partial\chi}{\partial z_j}\phi+\kappa\chi\frac{\partial\phi}{\partial z_j}\right)$$
$$= \oint_C\cdots\oint_C \mathrm{d}t_1\cdots\mathrm{d}\mu_M\left(\chi E_j^b\phi+\kappa\chi\frac{\partial\phi}{\partial z_j}\right)$$
$$= \oint_C\cdots\oint_C \mathrm{d}\mu_1\cdots\mathrm{d}\mu_M\left[\chi H_j^b\phi+\chi\sum_{\alpha=1}^{M}W(\mu_\alpha,z_j)f_\alpha E_j^-(d_\alpha)\phi_\alpha^b\right.$$
$$\left.+\kappa\chi\sum_\alpha\frac{\partial}{\partial\mu_\alpha}\left(W(\mu_\alpha,z_j)\phi_\alpha^b E_j^-(d_\alpha)\right)\right]$$
$$= \oint_C\cdots\oint_C \mathrm{d}\mu_1\cdots\mathrm{d}\mu_M\left[\chi H_j^b\phi+\kappa\sum_\alpha\frac{\partial}{\partial\mu_\alpha}\left(\chi W(\mu_\alpha,z_j)\phi_\alpha^b E_j^-(d_\alpha)\right)\right]$$
$$= H_j^b\Psi \tag{5.5}$$

where

$$W(\mu_\alpha,z) = \frac{(-1)^{\alpha-1}2\cosh(z_j)\sinh(\mu_\alpha)}{\sinh(\mu_\alpha+z_j)\sinh(\mu_\alpha-z_j)}.$$

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